

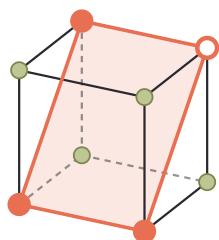


The maths of Møbee

Behind the deceptively simple game Møbee lies some serious mathematics: the finite Möbius plane.

Order (q)	Cube	3	4	5
Symbols per card $q(q+1)$	7	12	20	30
Cards in deck (n) q^2+1	8	10	17	26
Symbols in deck $q(q^2+1)$	14	30	68	130
Cards each symbol appears on (k) $q+1$	4	4	5	6
Shared by any 2 cards $q+1$	3	4	5	6
Shared by any 3 cards	1	1	1	1

7 symbols · the cube



The 7-symbol deck: 8 cards on the corners of a cube, 14 symbols. Each symbol is a set of 4 corners: one of the 6 faces, one of the 6 diagonal rectangles, or one of the 2 tetrahedra. Any three corners belong to exactly one of these sets.

- Three cards
- The fourth card with the same symbol
- The other four cards

WHY THE CUBE WORKS

Give every corner of the cube three coordinates, each 0 or 1, for example (0, 1, 1). The eight corners are the eight cards. A symbol is one of 14 special sets of four corners: the 6 faces, the 6 diagonal rectangles that cut through the cube, and the 2 tetrahedra you get by taking every other corner. Every corner belongs to 7 of these sets, so every card carries 7 symbols.

To find the fourth corner, add the three coordinates position by position and keep only odd or even: 1 if the sum is odd, 0 if it is even. For (0, 0, 0), (1, 0, 0) and (0, 1, 1) that gives (1, 1, 1). Any three corners give exactly one fourth, so **any three cards share exactly one symbol**. Any two cards share 3 symbols: the six other corners complete them, in pairs, to three sets of four.

For mathematicians

Every Møbee deck is a Steiner system $S(3, k, n)$: n cards, every symbol on k of them, and any 3 cards share exactly one symbol. The 7-symbol deck is one too, $S(3, 4, 8)$, but it is not a Möbius plane.

The eight corners with coordinates 0 and 1 form the affine space $AG(3, 2)$, three-dimensional geometry over the field with two elements. Its planes are exactly the 14 sets of four above, and any three points lie in exactly one plane.

12, 20 and 30 symbols · Möbius planes

WHAT IS A MÖBIUS PLANE?

Think of the surface of a ball. Any three points on it fix exactly one flat slice through the ball, and that slice cuts the surface in a circle. So **through any three points there is exactly one circle**. A Möbius plane is any collection of points and “circles” that obeys this rule; the circles don’t have to be round. It is named after August Ferdinand Möbius, who studied the geometry of circles.

FROM CIRCLES TO CARDS

Make every point a card and every circle a symbol: a card shows a symbol when its point lies on that circle. “Three points, exactly one circle” becomes **“three cards, exactly one shared symbol”**. Two points lie on $q+1$ circles, which is why any two cards share several symbols. The 7-symbol deck uses the same idea with a cube: any three of its eight corners lie in exactly one set of four.

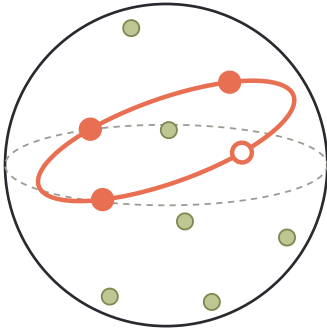
For mathematicians

A Möbius plane is defined by three rules. (1) Any three points lie on exactly one circle. (2) Given a circle, a point on it and a point off it, exactly one circle passes through both points and touches the first circle only at that point. (3) Every circle has at least three points, and not all points lie on one circle.

Finite Möbius planes exist too: for every prime power q (2, 3, 4, 5, 7, 8, 9, ...) there is a Möbius plane of order q . It has q^2+1 points, $q+1$ points on every circle, and $q(q+1)$ circles through every point.

In Møbee, a point is a card and a circle is a symbol. So every deck is a Steiner system $S(3, k, n)$: n cards, every symbol on k of them, and any 3 cards share exactly one symbol. The 12-, 20- and 30-symbol decks are the Möbius planes of order 3, 4 and 5: $S(3, 4, 10)$, $S(3, 5, 17)$ and $S(3, 6, 26)$.

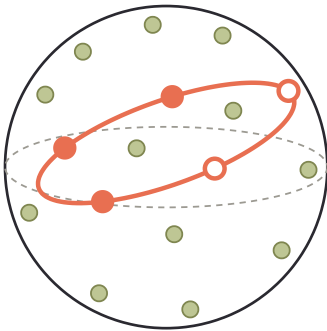
Order 3 · 12 symbols



Möbius plane of order 3: 10 cards, 12 symbols per card, 30 different symbols. Each circle is one symbol and passes through the 4 cards that show it. Any three cards lie on exactly one circle.

- Three cards
- The fourth card with the same symbol
- The other 6 cards

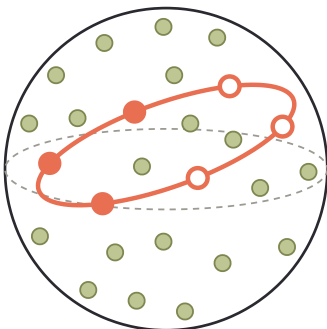
Order 4 · 20 symbols



Möbius plane of order 4: 17 cards, 20 symbols per card, 68 different symbols. Each circle is one symbol and passes through the 5 cards that show it. Any three cards lie on exactly one circle.

- Three cards
- The other 2 cards with the same symbol
- The other 12 cards

Order 5 · 30 symbols



Möbius plane of order 5: 26 cards, 30 symbols per card, 130 different symbols. Each circle is one symbol and passes through the 6 cards that show it. Any three cards lie on exactly one circle.

- Three cards
- The other 3 cards with the same symbol
- The other 20 cards

Further reading



Möbius Planes, Block Designs, and the Møbee Game

János Ruff · Érintő 2024/1 · English translation

<https://play.mobeecards.com/maths/mobius-planes-block-designs-mobee-game>

- Wikipedia: Steiner system: https://en.wikipedia.org/wiki/Steiner_system
- Wikipedia: Möbius plane: https://en.wikipedia.org/wiki/M%C3%B6bius_plane